

# Passive Tracking and Information Theory

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## ABSTRACT

Estimation of kinematic attributes using passive sensors is a key problem in air and missile defense as well as anti-submarine warfare and airborne surveillance. Traditional deterministic observability criteria are based on engagement and observation kinematics, and imply infinite tracking times and noise-free measurements. Real systems have noise corrupted measurements and finite tracking times or finite numbers of measurements. These "real world" constraints may be accommodated using information-theoretic concepts. Thus criteria may be formulated to ensure the necessary and sufficient conditions for unique estimates. These criteria are based on metrics which take into account noisy and finite numbers of measurements, and provide a basis for a definition of "stochastic observability". The intent of this paper is to introduce the deterministic criteria, develop a modification using information-theoretic distance measures to form a stochastic observability criteria, and finally to illustrate this development via simple examples.

## 1. Introduction

Estimation of a target trajectory using angle-only measurements is a critical problem in satellite tracking, ballistic missile defense, homing missile guidance, antisubmarine warfare, and airborne surveillance. The key to solving this problem, in general, and for solutions to these diverse applications, in particular, is defining target observability criteria: the necessary and sufficient conditions to ensure a unique estimate of the target trajectory given a finite set of noise-corrupted observations. The intent of this paper is to apply the information-theoretic observability measure previously developed in [1].

The earliest examples of trajectory estimation from angle-only measurements are the efforts of the Greeks (e.g., Ptolemy) to develop their models of planetary motion; these efforts demonstrate the impossibility of long-term trajectory prediction and estimation when the underlying dynamics models are wrong. Probably the modern era of mathematically-sound trajectory estimation based on a limited set of

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observations opened in 1801 with Gauss estimating the orbit of the asteroid Ceres using the newly-discovered "least squares" technique [2].

Since Gauss' time, the batch least-squares technique has been augmented by Kalman Filter techniques and the applications extended to homing missile guidance [3], antisubmarine warfare [4], and airborne surveillance [5]. The tracking community has long recognized the lack of observability inherent in angle-only track, and various techniques have been developed to compensate, e.g., Kalman Filter coordinate systems which decouple the range (unobservable) dynamics from the azimuth and elevation axes [6]. Recent efforts have focused on developing recursive finite dimensional filters as an approximation to the conditional target position probability density [7].

What is needed, of course, is a general theory of observability defining the conditions under which we could, in principle, estimate the target states. One would then know that if the observability conditions are violated in any given tracking scenario, development of special coordinate systems, numerical techniques, or nonlinear optimal estimators will not be successful. Thus, universal observability criteria could be extremely useful in developing system requirements and feasible system designs.

Section 2 reviews a deterministic model for developing observability conditions, discusses the model limitations, and illustrates these conditions via examples. Section 3 extends the observability conditions to include noise corrupted measurements and proposes information-theoretic observability measures.

## 2. Deterministic-Observability

For linear discrete systems the n-state Markov model takes the form:

$$x(t+1) = Ax(t) + Bu(t); x(0) = x_0 \quad (1)$$

$$Y(t) = Cx(t) \quad (2)$$

A particular state  $x$  is unobservable if for some  $\tau$  the initial state  $x_0 = x$  produces zero observations:

$$y(t) = 0 \quad 0 \leq t \leq \tau \quad (3)$$

If the observability matrix  $\Pi$

$$\Pi = \begin{bmatrix} C \\ CA \\ \bullet \\ \bullet \\ CA^{n-1} \end{bmatrix} \quad (4)$$

is of rank  $n$ , then the system [Eq (1) - (2)] is completely observable. This reasoning was used in [8] to develop 3-dimensional observability requirements. However, an alternative analysis which develops both necessary and sufficient conditions is the point of departure for this work [9].

The following paragraph summarizes the results presented in [9]. Consider the observer target geometry shown in Figure 1. The observer platform, P, has a position history  $\mathbf{w}(t)$ ; the target T, has a target history  $\mathbf{s}(t)$  and the relative position is:

$$\mathbf{r}(t) = \mathbf{s}(t) - \mathbf{w}(t) \quad (5)$$

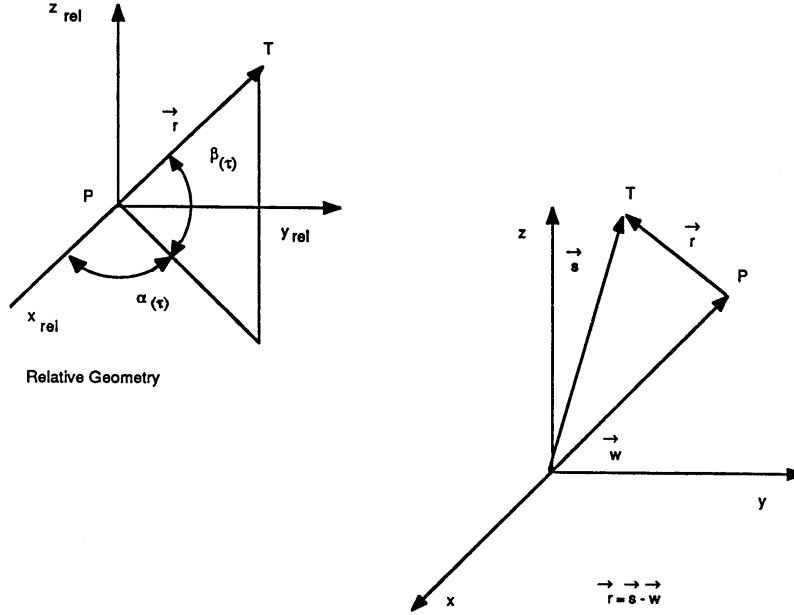


Figure 1 Definition of Relative Engagement Geometry

Typically, the observer measures azimuth,  $\alpha(t)$ , and elevation,  $\beta(t)$ , to the target. This measurement is equivalent to measuring the line-of-sight unit vector

$$\mathbf{u}(t) = \mathbf{r}(t) / |\mathbf{r}(t)| \quad (6)$$

Assume that the target motion can be modeled by Nth order dynamics

$$\mathbf{s}(t) = \mathbf{S} \mathbf{t} \quad (7)$$

where  $\mathbf{t} = [1 \ t \ t^2 \ \dots \ t^n]^T$  and  $\mathbf{S}$  is a  $3 \times (N+1)$  matrix of unknown coefficients which are to be estimated. This assumption is not as restrictive as it might first appear, since even very nonlinear trajectories such as that of boosting, staging, rockets can be modeled in by constant coefficient polynomials [9].

The observer's trajectory is defined as

$$\mathbf{w}(t) = \mathbf{W} \mathbf{t} - \mathbf{g}(t) \quad (8)$$

where  $W$  is the  $3 \times (N+1)$  matrix describing the observer's  $N$ th order dynamics and

$$\mathbf{g}(t) = [g_1(t), g_2(t), g_3(t)]^T$$

represents the observer's maneuver. The significance of the maneuver is that it does not contain power of  $t$  less than  $N+1$ .

Equations (5), (7) and (8) yield

$$\mathbf{r}(t) = \mathbf{R}t + \mathbf{g}(t) \quad (9)$$

where  $\mathbf{R} = \mathbf{S} - \mathbf{W}$ ; estimating the target trajectory is equivalent to estimating  $\mathbf{R}$ .

Further, from Equation (6) we have

$$\mathbf{r}(t) = |\mathbf{r}(t)| \mathbf{u}(t) = \lambda(t) \mathbf{u}(t) \quad (10)$$

Combining (9) and (10) yields

$$\mathbf{R}t + \mathbf{g}(t) = \lambda(t) \mathbf{u}(t) \quad (11)$$

Based on this analysis, the necessary and sufficient condition for observability is that

$$\mathbf{r}(t) \neq \gamma(t) \mathbf{A}t \quad (12)$$

where  $\gamma(t)$  is an arbitrary scalar function and  $\mathbf{A}$  is a  $3 \times (N+1)$  coefficient matrix.

Eq (11) may be interpreted as meaning that:

- A. The observer must have  $(N+1)$ st order dynamics when the target has  $N$ th order dynamics.
- B. The higher order dynamics must be orthogonal to the line-of-sight pointing vector between observer and target.

While this analysis is correct for the assumed model, the inclusion of measurement noise is critical to developing a practical theory of observability. The next section proposes an observability measure which incorporates measurement noise.

### 3. Information-Theoretic Observability Measures

The analysis that led to the observability conditions of Eq (11) was based on deterministic state dynamics  $W(t)$ ,  $S(t)$ , and the resulting  $\mathbf{r}(t)$ . These state trajectories could be classified as either observable or unobservable. In contrast to that binary situation, one would expect observability to be characterized by a continuum of values ranging from totally unobservable to totally observable. Further, intuition suggests that noise should also be taken into account; trajectories of almost unobservable targets estimated from noisy measurements should be, in some sense, distinguished from the same trajectories estimated from noiseless measurements. An approach to developing a concept of observability embodying these features

was presented in [11]. The concept relied on entropy as a measure of observability, which requires that the initial system states be assigned a probability measure. Implementation entailed using an artifact, that of randomizing the initial states.

To get around this difficulty, I propose that information-theoretic criteria used to measure class separability be employed here to measure the distance between the observable and the unobservable condition. One measure, among the many possible candidates, is the Bhattacharyya distance.

The Bhattacharyya distance,  $\mu_\beta$ , provides upper and lower error bounds on discrimination when using a Bayes classifier [12]. For normal distributions

$$\mu_\beta = \frac{1}{8} (M_1 - M_2)^T \left[ \frac{\Sigma_1 + \Sigma_2}{2} \right]^{-1} (M_1 + M_2) + \frac{1}{2} \ln \frac{|\Sigma_1 + \Sigma_2|}{|\Sigma_1|^{\frac{1}{2}} |\Sigma_2|^{\frac{1}{2}}} \quad (13)$$

where  $M_i$ ,  $\Sigma_i$ ,  $i = 1, 2$  are the means and covariances of the class distribution. For this problem, the two classes of interest are the observable and the unobservable trajectories; one option for the measurable quantities which can be used to represent the two classes are the angle histories, ( $\Theta$  (t) for the examples of section 2.0) and then  $\Sigma_i$  are the respective measurement covariances. Using the results of Section 2.0 and assuming that the measurement covariances are equal simplifies Eq (13) to

$$\mu_\beta = \frac{1}{8} \frac{(\Delta\Theta)^2}{\Sigma} \quad (14)$$

where

$$\Delta\Theta = \Theta_i - \Theta_j$$

$\Sigma$  = measurement covariance

Equation (14) has the obvious interpretation of being a signal-to-noise ratio where the signal is the angular difference between the observable and unobservable trajectories.

## 4. Bhattacharyya Distance Example

Figure 2 depicts a tracking engagement in which the observer aircraft is flying east at 350 knots while the target is heading west at 600 knots. Figure 2 shows the observer maneuvering by flying a repeating sequence of left and right  $\frac{1}{4}$  g' turns, each turn lasting approximately 100 seconds, providing the higher order dynamics. Figure 3 is the bearing history for the maneuvering platform. In addition, coplotted with the accelerating observer, is the bearing history of the same observer flying a constant velocity trajectory. The difference in bearing histories is shown in Figure 4.

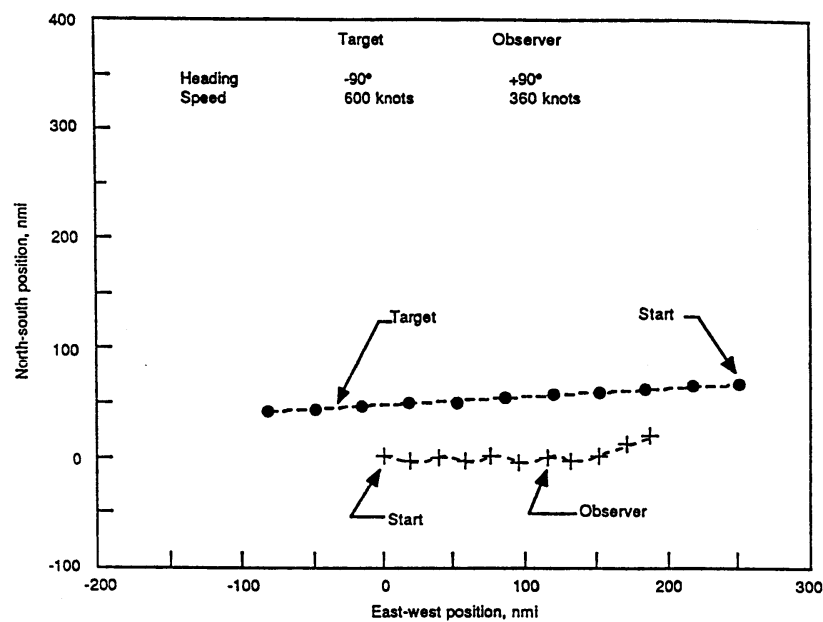


Figure 2 Engagement Geometry for Passive (Angle-Only-Track) Bhattacharyya Measure Example

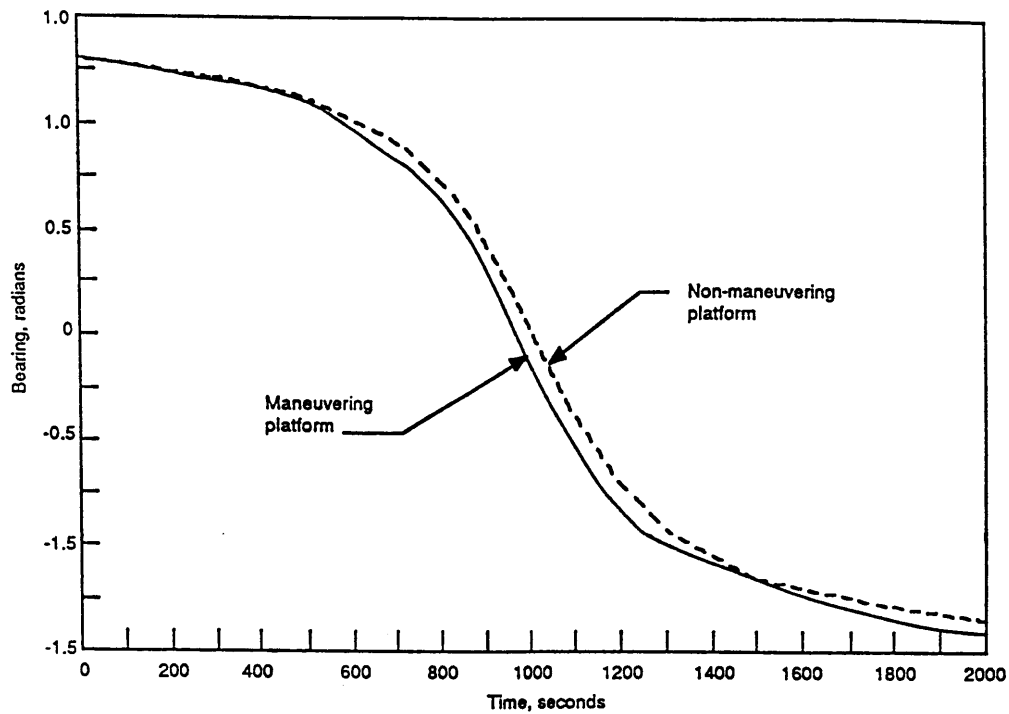


Figure 3 Bearing History of Example Tracking Equipment

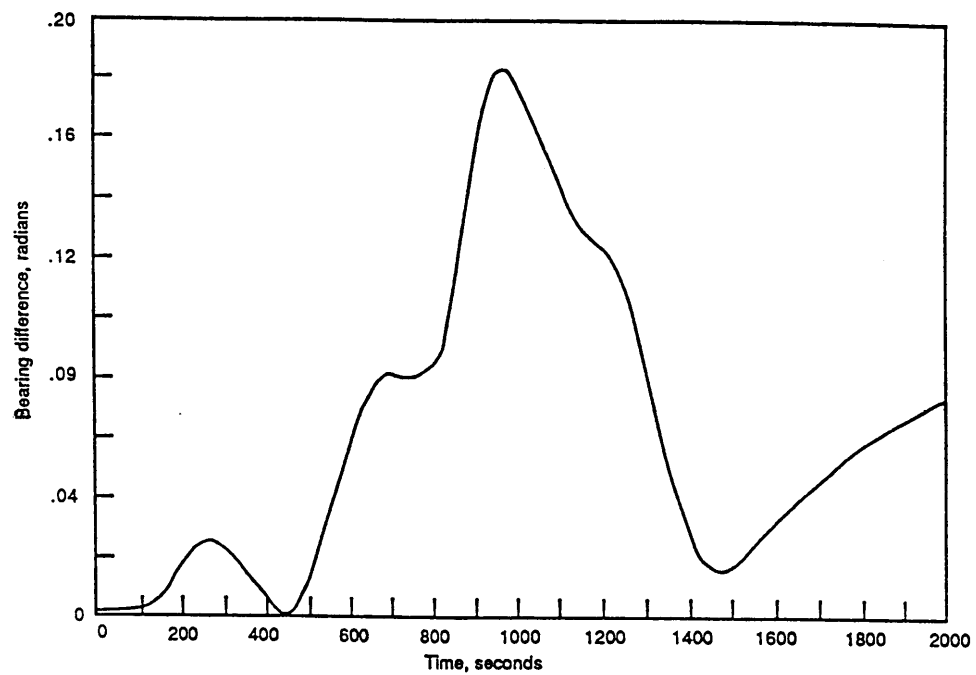


Figure 4 Bearing History Difference Between Maneuvering and Non-Maneuvering Platforms

A simulation testbed modeling this engagement, simulating noisy bearings measurements, and estimating range via a modified polar coordinate system filter, was utilized to characterize the estimator performance and relationship to the Bhattacharyya distance. For this engagement, Figure 5 presents the Bhattacharyya distance,  $\mu_\beta$  as a function of track time, parameterized by angular measurement accuracy. As expected, high measurement noise leads to lower Bhattacharyya distances.

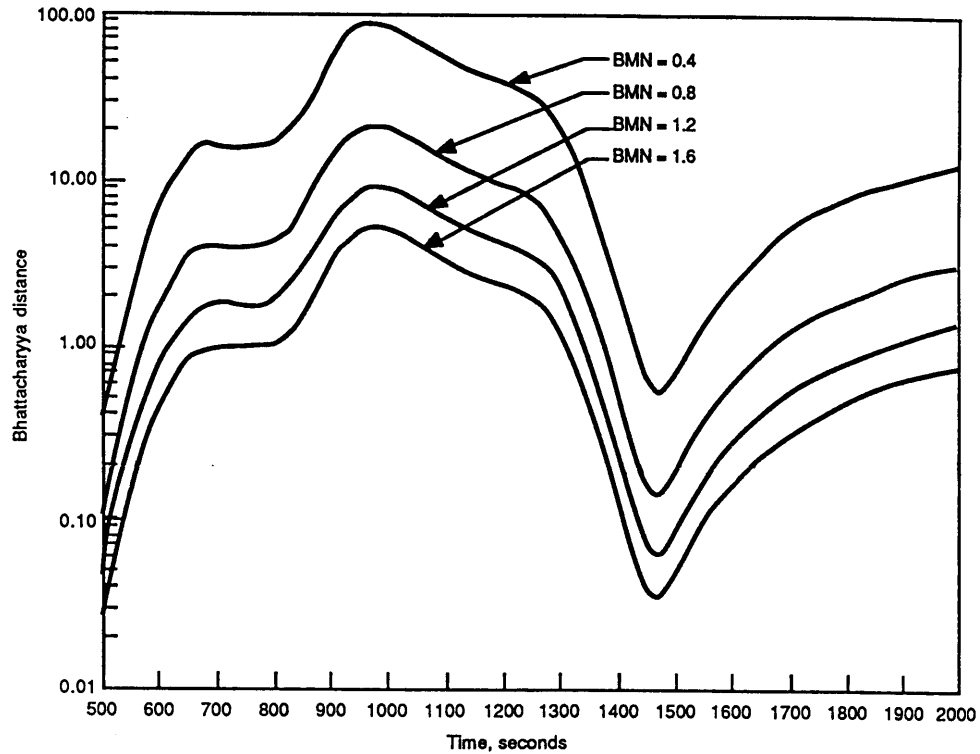
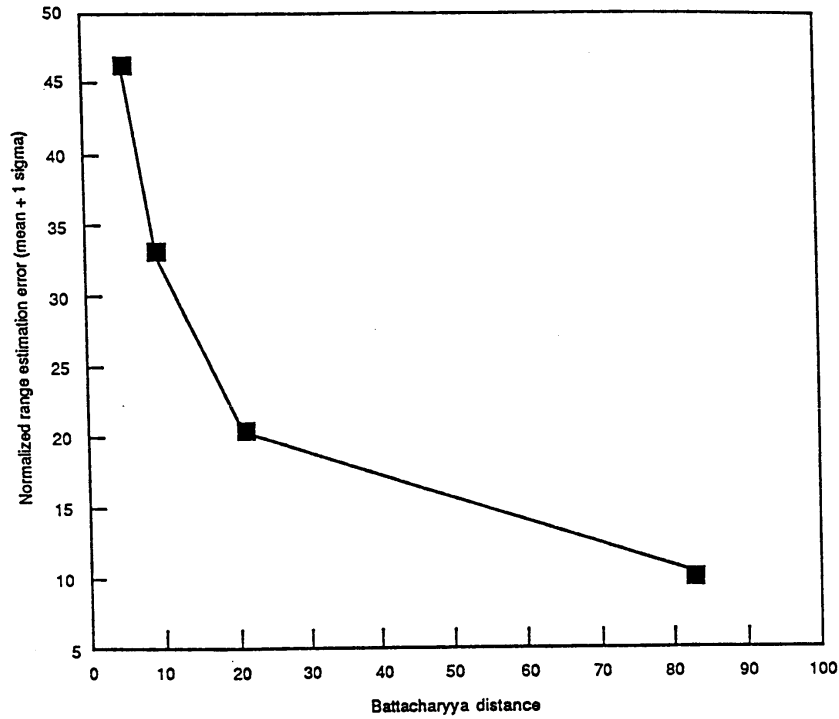


Figure 5 Bhattacharyya Distance, Parameterized by Bearing Measurement Noise (BMN), as a Function of Track Time

What is of interest, of course, is the relationship between the estimator performance and  $\mu_\beta$ . Using a modified polar coordinates Kalman Filter and a Monte Carlo simulation of the noisy bearing measurements, the range error as a function of Bhattacharyya distance was estimated. Figure 6 presents the normalized range error versus the Bhattacharyya distance (all evaluated at a track time of 1000 seconds). As shown in Figure 6 accurate estimator performance (lower normalized error) is strongly correlated with higher  $\mu_\beta$ . This result satisfies the intuition that large  $\mu_\beta$  is a predictor of accurate estimator performance.



*Figure 6 Normalized Range Error, After 100 Seconds of Tracking, as a Function of Bhattacharyya Distance*

## **5. Conclusions and Future Investigations**

A stochastic observability criteria based on the Bhattacharyya distance has been developed and demonstrated via an example. The performance of a sophisticated nonlinear estimator, a passive range polar coordinate filter, has been correlated with the Bhattacharyya distance. As hoped, the larger the observability measure, the better (more accurate) the estimator performance.

Future investigations will be directed at characterizing the relationship between estimator performance and observability measures.

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